Two-Dimensional Diffusion Wave Model for Numerical Simulation of Inundation – Upper Narew Case Study

Michał SZYDŁOWSKI
Faculty of Civil and Environmental Engineering
Gdańsk University of Technology
ul. Narutowicza 11/12, 80-952 Gdańsk, Poland
e-mail: mszyd@pg.gda.pl

Abstract
The purpose of this paper is to present the numerical calculations which can be useful for simulations of inundations in natural river valleys. When estimating the reach and area of the inundation related to river flow, digital elevation model and mathematical model of flood wave propagation are indispensable. For numerical simulation of flood, the mathematical model of free surface unsteady water flow can be applied. Usually, the one- or two-dimensional shallow water equations, called Saint Venant equations, are assumed. In this paper, the simplified hydrodynamics model, known as a diffusion wave model, is presented and applied for simulation of inundation along the Upper Narew reach connecting Suraż and Łapy. The model equations are solved using finite volume method.

Key words: mathematical modelling, diffusion wave model, finite volume method, floodplain inundation, flood zones, Upper Narew.

1. Introduction
Nowadays, numerical simulations of river inundations are the basic tool for flood risk mitigation and water management in the river catchments. The water flow prediction is indispensable not only for management of risk resulting from floods but can be useful for irrigation problems as well. Therefore, estimation of water flow parameters in rivers and floodplains is one of the major tasks for water engineers and hydrologists.

The Upper Narew reach, analyzed in this case study, is crossing the Narew National Park. In this region, floods are very important element of natural environment. There are even some concepts of artificial inundations in this region (controlled at Siemianówka reservoir) sustaining animal life and appropriate moisture levels in the soil (Rowiński et al. 2005). The hydraulic consequences of such flood waves can be simulated using mathematical models too.
When estimating the water flow parameters (depth and velocity) mathematical model to describe the free surface flow is needed. The models of water flow dynamics can range in complexity from simple one-dimensional Bernoulli equation for steady flow to full three-dimensional solutions of the Navier-Stokes equations with some turbulence models (Sawicki 1998). Unfortunately, the latter is still too complex to be applied for practical cases (Szydłowski and Zima 2006).

Recently, the most widely used models for free surface water flow modeling have been Saint Venant equations for one-dimensional open channel flow. Numerical solution of the equations provides the user with the cross-section-averaged velocity and water depth at each location. In order to estimate the inundation zone the values of water surface level can be interpolated between river (valley) cross-sections. The one-dimensional hydrodynamic models are usually efficient for modeling of water wave propagation in open channels, but they are often inconvenient for inundation prediction due to simplified representation of floodplain areas. When the flow area is defined only with some cross-sections of river channel and floodplain it can result with some errors in the inundation zone prediction. In order to overcome this problem two-dimensional models should be applied, especially for floodplain flows. They ensure the water level and averaged velocity to be computed at each computational node of numerical grid representing the flow area. Two-dimensional shallow water equations, called two-dimensional Saint Venant equations or briefly dynamic wave model) can be obtained from the Navier-Stokes equations using a depth averaging procedure (Szymkiewicz 2000). For this hydrodynamic model it is assumed that vertical component of velocity can be neglected, pressure field is hydrostatic, bottom slope is small and bottom friction can be approximated as for a steady flow.

This model can be further simplified by reducing description of some physical processes present in the water flow phenomenon. In this paper the inundation of the floodplains due to Narew flow is investigated using simplified model of two-dimensional free surface water flow, named the diffusion wave model. In order to solve the equations, numerical scheme based on finite volume method is applied.

2. Governing equations

The two-dimensional dynamic wave model can be presented as the following system of partial differential equations (Tan 1992), representing mass and momentum conservation:

\[ \frac{\partial H}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0, \] (1a)

\[ \frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x}\left( \frac{q_x^2}{h} \right) + \frac{\partial}{\partial y}\left( \frac{q_x q_y}{h} \right) + gh \frac{\partial H}{\partial x} + gh S_{fx} = 0, \] (1b)

\[ \frac{\partial q_y}{\partial t} + \frac{\partial}{\partial x}\left( \frac{q_x q_y}{h} \right) + \frac{\partial}{\partial y}\left( \frac{q_y^2}{h} \right) + gh \frac{\partial H}{\partial y} + gh S_{fy} = 0. \] (1c)
In this system of equations $H$ and $h$ represent the water surface level and depth, $q_x$ and $q_y$ are the flow rates in $x$ and $y$ direction, respectively. $S_{fx}$ and $S_{fy}$ are the bottom friction terms, $g$ is the acceleration due to gravity and $t$ represents time. The bottom friction can be defined using the Manning formula (Tan 1992), for example:

$S_{fx} = \frac{n^2 \cdot q_x \cdot \sqrt{q_x^2 + q_y^2}}{H^{10/3}}$, \hspace{1cm} (2a)$

$S_{fy} = \frac{n^2 \cdot q_y \cdot \sqrt{q_x^2 + q_y^2}}{H^{10/3}}$, \hspace{1cm} (2b)$

where $n$ denotes the Manning friction coefficient.

To solve the shallow water equations, relatively complicated numerical methods and complex data set must be used. To reduce the mathematical complexity of the model some terms in the model can be neglected, but only when it is reasonable. Making appropriate assumption (Szymkiewicz 2000) one can find: diffusion wave model (non-inertial model), kinematic wave model, and flat pond model (if only mass conservation is considered).

The two-dimensional diffusion wave model, used here for numerical simulation of flood inundation of the Upper Narew floodplain, can be obtained following Hromadka and Yen’s (1987) idea. For the sake of clarity, only one equation of motion (in the $x$ direction, for example) can be considered. The local and convective acceleration terms can be grouped and Eq. (1b) can be rewritten as:

$$\frac{1}{gh} \left[ \frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q_x^2}{h} \right) + \frac{\partial}{\partial y} \left( \frac{q_x q_y}{h} \right) \right] = -\left( \frac{\partial H}{\partial x} + S_{fx} \right).$$ \hspace{1cm} (3)$

The same equation can be presented as a short formula:

$$S_{fx} = -\left( \frac{\partial H}{\partial x} + m_x \right).$$ \hspace{1cm} (4)$

If it is assumed that the friction slope is approximated using Manning equation for steady flow (Eq. 2a) it is possible to include this empirical formula to the equation of motion (Eq. 1b). After that, one can obtain:

$$\frac{n^2 \cdot q_x \cdot q_x}{H^{10/3}} = -\left( \frac{\partial H}{\partial x} + m_x \right),$$ \hspace{1cm} (5)$

where subscript $s$ indicates the flow direction which makes an angle $\phi$ with the $x$ direction, giving:

$$q_s = q_x \cdot \cos \phi.$$ \hspace{1cm} (6)$

Finally, after arrangement of Eqs. 5 and 6, the flow equation in the $x$ direction can be written as:
\[ q_x = -K_x \frac{\partial H}{\partial x} - K_x \cdot m_x, \quad (7) \]

where

\[ K_x = \frac{\frac{1}{h^{5/3}}}{\left[ \frac{\partial H}{\partial s} + m_y \right]^{\frac{1}{3}}} \quad (8) \]

is the hydraulic conduction parameter in the \( x \) direction, which is limited in value by the dominator term being checked for a small allowable amount (Singh 1996). It should be underlined here that the assumption about the absolute value of this term seems to be limiting the hydrodynamics model and its influence on numerical simulation of water flow should be investigated in details. In a similar way, the flow in the \( y \) direction can be presented as:

\[ q_y = -K_y \frac{\partial H}{\partial y} - K_y \cdot m_y. \quad (9) \]

Putting the flow rates formulations (Eqs. 8 and 9) into the continuity Eq. (1a) gives the following relationship:

\[ \frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left( K_x \frac{\partial H}{\partial x} \right) - \frac{\partial}{\partial y}\left( K_y \frac{\partial H}{\partial y} \right) - S = 0, \quad (10) \]

where

\[ S = \frac{\partial}{\partial t} K_x m_x + \frac{\partial}{\partial y} K_y m_y. \quad (11) \]

The final form of hydrodynamics model depends on the parameters \( m_x \) and \( m_y \), which represent inertia forces in \( x \) and \( y \) directions as terms of local and convective acceleration (Eq. 3). The number of acceleration terms is related to the model simplification. Various forms of \( m_x \) can be assumed, for example:

\[ m_x = \frac{1}{gh} \left[ \frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q_x^2}{h} \right) + \frac{\partial}{\partial x} \left( \frac{q_y q_x}{h} \right) \right], \quad (12a) \]

\[ m_x = \frac{1}{gh} \left[ \frac{\partial}{\partial x} \left( \frac{q_x^2}{h} \right) + \frac{\partial}{\partial x} \left( \frac{q_y q_x}{h} \right) \right], \quad (12b) \]

\[ m_x = \frac{1}{gh} \left[ \frac{\partial q_x}{\partial t} \right], \quad (12c) \]

\[ m_x = 0. \quad (12d) \]

If all acceleration terms are present in the parameter \( m_x \) (Eq. 12a) and \( m_y \), the hydrodynamics model is full dynamic wave model written in other form than shallow water.
equations (Eq. 1). Assuming \( m_x = m_y = 0 \), which means neglecting inertia forces terms in the flow description, one obtains a two-dimensional diffusion wave model, which can be presented in one-equation form as follows:

\[
\frac{\partial H}{\partial t} - \frac{\partial}{\partial x} \left( K_x \frac{\partial H}{\partial x} \right) - \frac{\partial}{\partial y} \left( K_y \frac{\partial H}{\partial y} \right) = 0.
\]  

(13)

3. Solution method

Solution of diffusion wave model (Eq. 13) can be realized using any method of numerical integration of partial differential equation in space and time. In this paper the finite volume method (LeVeque 2002) is proposed for model space approximation. For time integration the second order Runge-Kutta (Tan 1992) scheme is used.

To integrate Eq. (13) in space using finite volume method the calculation domain can be discretized into a set of triangular cells (Fig. 1), for example. This kind of approximation ensures an unstructured numerical mesh. Each cell is defined by its centre point and variable \( H \) is averaged and constant inside the cell.

![FVM discretization of calculation domain.](image_url)

Equation (13) can be rewritten in the following form:

\[
\frac{\partial H}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0,
\]  

(14)

where

\[
F = -K_x \frac{\partial H}{\partial x}, \quad G = -K_y \frac{\partial H}{\partial y},
\]  

(15)

or shorter as:

\[
\frac{\partial H}{\partial t} + \nabla \cdot D \mathbf{n} = 0,
\]  

(16)

where
and \(\mathbf{n} = (n_x, n_y)^T\) is an unit vector.

After integration and applying the Ostrogradski-Gauss theorem for each cell \(i\), Eq. (16) can be written as:

\[
\frac{\partial H}{\partial t} \frac{\Delta A_i}{\Delta t} + \oint_{L_i} \mathbf{D} \mathbf{n} \, dL = 0, \tag{18}
\]

where \(\Delta A_i\) and \(L_i\) are the area and boundary of cell \(i\). The first term of Eq. (18) represents time evolution of water surface level over cell \(i\), which is related to the water volume change inside the cell. The second one is the flow rate (discharge) through the boundary of cell \(i\). This flux is of diffusive type. The integral in Eq. (18) can be substituted by a sum of three components as follows:

\[
\frac{\partial H}{\partial t} \frac{\Delta A_i}{\Delta t} + \sum_{r=1}^{3} D_r \mathbf{n} \cdot \Delta L_r = 0, \tag{19}
\]

where \(D_r\) is the flux computed at the \(r\)th cell-interface, and \(\Delta L_r\) represents the cell-interface length. In order to simulate the free surface water flow, Eq. (19) must be solved inside every finite volume \(i\). Therefore, the main point of the solution technique is an approximation of fluxes between computational cells.

In order to estimate the fluxes \(D_r\), derivative of water surface level \(H\) at the \(r\)th cell-interface is required. For two cells, \(i\) and \(ii\) (Fig. 2), the derivative can be approximated by weighted average of derivatives computed inside the triangles \(T_1\) (vertexes \(i, j, k\)) and \(T_2\) (vertexes \(j, ii, k\)).

![Fig. 2. Graphic scheme for \(H\) derivative calculation.](image)

For the sake of clarity, only the derivative in the \(x\) direction, \(H_x = \partial H / \partial x\), is analyzed. If the derivatives in triangles \(T_1\) and \(T_2\) \((H_x^{T_1}, H_x^{T_2})\) are known, the averaged value at the \(r\)th cell-interface can be expressed as:
where $A^{T1}$ and $A^{T2}$ are the areas of triangles $T1$ and $T2$, respectively. To compute $H'_s$, the values of derivatives inside both triangles are needed. They can be calculated like for finite element method with linear base functions (Zienkiewicz 1972). The values of $H_s$ and $H_t$ in triangle $T1(i, j, k)$ can be approximated as:

$$H'_s = \frac{H^{T1}_s A^{T1} + H^{T2}_s A^{T2}}{A^{T1} + A^{T2}},$$

(20)

(21)

(22)

where $H_s$, $H_t$ and $H_k$ are the values of water surface level at nodes $i$, $j$, and $k$ respectively. The factors $b_l$ and $c_l$ ($l = i, j, k$) are

$$b_l = y_j - y_k, \quad b_j = y_k - y_i, \quad b_k = y_i - y_j,$$

(23)

$$c_l = x_k - x_j, \quad c_j = x_j - x_k, \quad c_k = x_k - x_i,$$

(24)

where $x_l$ and $y_l$ ($l = i, j, k$) are the coordinates of the vertexes $i$, $j$, $k$. Unfortunately, considering the triangle $T1$, the water surface level is known only at node $i$ (inside cell $i$). At vertexes $j$ and $k$ the $H$ values must be approximated. They can be obtained by a distance weighted averaging of cell-centre values from cells surrounding the vertexes $j$ and $k$, respectively. For instance, vertex $j$ (Fig. 2) is included into six ($i$, $ii$, $iii$, $iv$, $v$, $vi$) neighboring cells and values of water surface levels from all these nodes must be taken into consideration. In a similar manner the derivatives in triangle $T2$ can be calculated. Finally, the flux at the $r$th cell-interface can be calculated from formulas (15) and (17).

4. Pilot numerical simulation of the Upper Narew inundation

In order to analyze the possibility to use the diffusion wave model to simulate and predict the floods in natural floodplains, the Upper Narew test case is presented here. The part of Upper Narew reach connecting Suraż and Łapy was considered. To make the two-dimensional numerical simulation of the flood flow on the real topography area the terrain information is necessary. It can be obtained from digital elevation model (DEM) – part of the digital terrain model (DTM). The model of relief of the Narew valley, applied in the computation, was prepared at the Institute of Geophysics of the Polish Academy of Sciences (Rowiński et al. 2005). The geometry of flow area and DEM of floodplain are presented in Fig. 3. The morphology of the river channel was not considered in the simulation due to coarse resolution of numerical mesh representing the flow area. To simulate the flooding the calculation domain was covered by unstructured triangular mesh composed of 14,421 computational cells of 50 m side length (Fig. 4). The mesh used in the numerical simulation was locally refined (25 m) in the vicinity of Suraż and Łapy bridges and along the valley axis. At the val-
ley side boundaries, in high lying areas, the coarse mesh was implemented (100 m). During the flow simulation the ground level is linearly interpolated inside each computational cell using DEM data. The constant value of Manning friction coefficient $n = 0.08 \, \text{m}^{-1/3} \, \text{s}$ in the whole flow area was used for the calculation reported in this paper.

Initially the whole flow domain was treated as a dry area (no water condition was imposed on the valley surface). At the open inflow boundary (the Suraż bridge cross-section) the water discharge was imposed as a boundary condition. The partial water rates through each cell-interface, making the inflow section up, were calculated splitting the total water discharge. At the outflow boundary (the Łapy bridge cross-section) the critical flow condition (critical water depth) was imposed. The rest of the boundaries were treated as closed boundaries. Due to relatively large dimensions of mesh elements, some details of the Narew valley relief were lost (system of the river channels, for example). The bridges were not taken into account during simulation. The calculation

![Fig. 3. DEM of the Upper Narew valley (Suraż–Łapy reach).](image)
was carried out with the time step, $\Delta t = 0.01$ s, ensuring the calculation stability. The total simulation time was 96 hours (4 days). This time is relatively short in relation to the duration of the real floods observed at the Suraż control cross-section. The floods recorded in the Upper Narew valley are usually longer than a month. The short time of simulation was a result of computation time step, which is limited with Courant-Friedrichs-Lewy (CLF) condition (Potter 1982) holding for explicit numerical schemes. Therefore, it was decided to simulate propagation of a test case flood wave only. The hydrograph of the test flood wave is presented in Fig. 5. At the beginning of the simulation the water discharge equal to 15 m$^3$/s was imposed at the inflow section; then, during first twelve hours, the water rate was increased up to 250 m$^3$/s. The constant maximum discharge has been imposed for one day. After this time it was decreased down to the base flow (i.e. 15 m$^3$/s) in the next 12 hours. The base and maximum values of the water rate were assumed to represent approximately the Narew mean flow and $Q_{\text{max}}$ of historical flood wave recorded at Suraż cross-section in 1979 (Rowiński et al. 2005).
Fig. 5. Test flood wave at the Suraż cross-section.

Fig. 6. Computed contours of water depth [m] after \( t = 12 \) h.
The results of Upper Narew flood test simulation are presented in Figs. 6, 7 and 8. In these figures, the evolution of flood zone in time can be investigated. The contours of water depth on the DEM background are shown there for three moments (simulation time steps) – $t = 12$ h, 48 h and 96 h after beginning of flood simulation. During simulation the flood wave propagates in many directions in the horizontal plane. It can be seen that the flood expands adequately to the relief of floodplain. The flow is split between the river channels along the Narew valley. The front of the flood wave, which propagates along the Narew valley, is reaching the outflow Łapy cross-section approximately after 24 hours. This time is longer than the historically observed time of the wave propagation along the Narew channel between Suraż and Łapy. This disagreement can be a result of the dry valley initial condition imposed here for two-dimensional flow modeling. Neglecting the base flow in the river channel and the channel itself due to coarse numerical mesh resolution has resulted with significant reduction of the front propagation velocity. In order to improve the calculation of the flooding time the complex – one-dimensional for the river flow and two-dimensional for floodplain inundation – simulation should be carried out.

Fig. 7. Computed contours of water depth [m] after $t = 48$ h.
In Fig. 7 the distribution of water depth after 48 hours of simulation is shown. It is almost the moment of maximum flood range in this pilot numerical simulation. The predicted inundation can be compared with the range of historical flood (1979) computed using one-dimensional hydrodynamic model (Rowiński et al. 2005). The flood zone simulated with two-dimensional diffusion wave model seems to be in accordance with the mentioned one, following the floodplain relief. Further detailed investigation of this historical flood is indispensable to verify the model.

After 48 h of simulation the discharge at the inflow boundary (Suraż) is reduced to the river base flow. It has resulted with the decreasing of the flood range. In Fig. 8 the isolines of water depth after 96 hours of simulation are presented. The inundation area is significantly smaller than before. Unfortunately, the time of the Narew valley draining in this area is longer than the simulation time. It has made the total valley emptying process impossible to simulate with the explicit numerical scheme.

![Computed contours of water depth [m] after t = 96 h.](image)

The data presented in the above-mentioned figures can be used as the background for flood mapping. The computed water depth is useful for risk identification and wa-
ter management in the river valleys. Such information can be also used for the analysis of the Upper Narew valley artificial inundations.

5. Conclusions

The pilot numerical simulation of inundation of the Upper Narew valley between Suraż and Łapy was presented in the paper. The two-dimensional diffusion wave model was proposed as a mathematical model of the free surface water flow. The equations of hydrodynamics were solved using finite volume method.

The results of numerical computation of the test inundation case have proved that the proposed mathematical model of the wave propagation in natural floodplains can be successfully applied for simulation of short time artificial flood in Upper Narew valley. It seems that diffusion wave model can be also useful for the analysis of practical (long time) inundation problems and water management in Upper Narew flow area. However, it is suggested to split mathematical modeling process into the one- and two-dimensional hydrodynamic models to separately simulate the flood wave propagation along the river and inundation of floodplain. Such a splitting technique can make the time of computations shorter ensuring longer periods of simulations.

Moreover, the detailed investigation of initial and boundary conditions is needed to properly simulate and foresee the hydraulic results of flood propagation in the Upper Narew valley. Finally, in order to confirm that the non-inertia hydrodynamics model is adequate to represent the real inundation problems in Upper Narew valley, a careful verification of the diffusion wave model, based on historical flood data is absolutely indispensable in future.

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